Mathematical Induction

Applications

1. Let $P(n)$ be the proposition : "The total number of segments, formed by n lines on a plane such that no two of them are parallel and no three of them are concurrent is n^2."

For $P(1)$, It is clear that the number of segments formed by 1 line is $1^2 = 1$.

Assume $P(k)$ is true for some $k \in \mathbf{N}$, that is,

the total number of segments formed by k lines is k^2 …. (1)

For $P(k + 1)$,

When the $(k + 1)^{th}$ line is inserted to the existing k lines (hence $\mathbf{k^2}$ segments by (1)), the maximum number of lines cut by the $(k + 1)^{th}$ line is $\mathbf{k}$ and also the $(k + 1)^{th}$ line is cut into $\mathbf{(k + 1)}$ more segments. $\therefore$ After insertion of the $(k + 1)^{th}$ line, the total number of segments

$$= k^2 + k + (k + 1) = (k + 1)^2 .$$

$\therefore$ $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall\ n \in \mathbf{N}$.

Let $P(n)$ be be the proposition : "The maximum number of sections cut by n lines is

$s_n = 1 + \frac{n}{2}(n+1)$, such that no two lines are parallel and no three of them are concurrent."

For $P(1)$, It is clear that the number of sections cut by 1 line is $s_1 = 1 + \frac{1}{2}(1+1) = 2$.

Assume $P(k)$ is true for some $k \in \mathbf{N}$, that is,

the max. number of sections cut by k lines is $s_k = 1 + \frac{k}{2}(k+1)$ …. (2)

For $P(k + 1)$, Suppose there are k lines existing, therefore the number of sections $= s_k$.

We now add a $(k + 1)^{th}$ line, this line cuts the k existing line, and $(k + 1)$ sections are formed.

$\therefore$ The total number of section $= s_k + (k + 1) = 1 + \frac{k}{2}(k+1) + (k+1) = 1 + \frac{k+1}{2}(k+2)$

$\therefore$ $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall\ n \in \mathbf{N}$.

2. Let $P(n)$ be the proposition : "two colours is enough to shade the regions divided by the circles so that no two adjacent regions is of the same colour."

For $P(1)$, Print the circle with black and outside the circle with white.

Assume $P(k)$ is true for some $k \in \mathbf{N}$, that is, two colours (black and white) is enough to shade the regions divided by the circles so that no two adjacent regions is of the same colour.

For $P(k + 1)$, Given, k+1 circles in the plane, throw out one circle, call it C, and by inductive hypothsis, colour with two colours the regions produced by the remaining k circles. Now put C back in. Some of the regions will be split in two, others will be unaffected. Leave the colours of all regions `outside' C unchanged and switch the colours of all regions `inside' C. This procedure produces a 2-colouring of the plane with k+1 circles.

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall$ n $\in$ **N**.

3. A simple n-sided polygon is divisible into exactly n-2 triangles by means of exactly n-3 non-intersecting diagonals. The result is the same for non-convex polygon. Proof omitted.

4. **(i)** Let the length of one side of the regular polygon be a_n.

Let $AB = a_n$, $AC = a_{2n}$, $OA = OB = r$

We like to found are relationship between a_n and a_{2n} first.

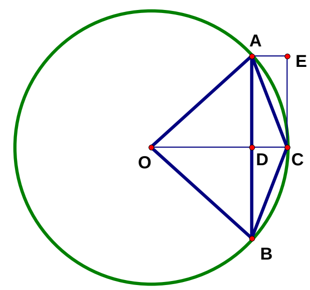

Diagram 1

$OD = \sqrt{OA^2 - AD^2} = \sqrt{r^2 - (a_n/2)^2}$

$DC = OC - OD = r - \sqrt{r^2 - (a_n/2)^2}$

$\therefore \quad a_{2n} = AC = \sqrt{AD^2 + DC^2} = \sqrt{(a_n/2)^2 + \left[r - \sqrt{r^2 - (a_n/2)^2}\right]^2}$

$= \sqrt{2r^2 - r\sqrt{4r^2 - a_n^2}}$ .... (1)

Let a_{2^n} be the length of one side of the regular 2^n-sided polygon.

Let P_{2^n} be the perimeter of the regular polygon

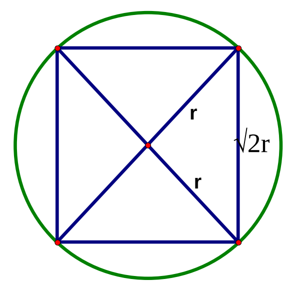

Diagram 2

From diagram 2, we get $a_{2^n} = \sqrt{2}r$, $P_{2^n} = 4\sqrt{2}r$

From (1), $a_{2^3} = a_8 = a_{2\times 4} = \sqrt{2r^2 - r\sqrt{4r^2 - (\sqrt{2}r)^2}} = \sqrt{2 - \sqrt{2}}\,r$

$P_{2^3} = 2^3 a_8 = 2^3\sqrt{2 - \sqrt{2}}\,r$

We like to use Mathematical Induction to prove that the proposition

P(n) : $a_{2^n} = \sqrt{2 - \sqrt{2 + \sqrt{2 - .... + (-1)^n\sqrt{2}}}}\,r$, $\quad P_{2^n} = 2^n a_{2^n}$ .... (2)

is true $\forall$ n $\in$ **N**, $n \geq 2$.

P(2) and P(3) are true from the above.

Assume P(k) is true for some k $\in$ **N** : $a_{2^k} = \sqrt{2 - \sqrt{2 + \sqrt{2 - .... + (-1)^k\sqrt{2}}}}\,r$, $\quad P_{2^k} = 2^k a_{2^k}$ .... (3)

We like to prove that the proposition is true for n = k + 1,

$a_{2^{k+1}} = a_{2\times 2^k} = \sqrt{2r^2 - r\sqrt{4r^2 - a_k^2}} = \sqrt{2r^2 - r\sqrt{4r^2 - \left(\sqrt{2 - \sqrt{2 + \sqrt{2 - .... + (-1)^k\sqrt{2}}}}\,r\right)^2}}$, by (3)

$= \sqrt{2 - \sqrt{2 + \sqrt{2 - .... + (-1)^k\sqrt{2}}}}\,r$ $\quad\therefore$ The proposition is true for n = k + 1.

By the Principle of Mathematical Induction, the proposition is true $\forall$ n $\in$ **N**, $n \geq 2$.

(ii) From (2), take r = 1, $\lim_{n\to\infty} P_{2^n}$ = length of the circumference = 2π

$$\lim_{n\to\infty} 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2-....+(-1)^n\sqrt{2}}}}}_{n-1\ \text{times}} = 2\pi$$

(iii) From (i), Take $r = 1$, $a_6 = 1$, $a_{12} = \sqrt{2-\sqrt{4-1}} = \sqrt{2-\sqrt{3}} = \frac{1}{2}\left(\sqrt{6}-\sqrt{2}\right)$

$a_{24} = \sqrt{2-\sqrt{2+\sqrt{3}}}$, $a_{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, $a_{96} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$

$$a_{6\times 2^k} = \sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2+\sqrt{3}}}}} \qquad .... \quad (4)$$

Let S_n be the area of the n-sided regular polygon. A be the area of the circle.

We can see that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{2n} = A$

In Diagram 1 above, sector AOC < ΔAOC + ΔAEC and ΔAEC = ΔADC = ΔAOC - ΔAOD

∴ Sector AOC < ΔAOC + (ΔAOV – (1/2)ΔAOB .... (5)

From (5) , multiply both sides by 2n, we have $A < S_{2n} + (S_{2n} - S_n)$.

Obviously $S_{2n} < A$, $\therefore\ S_{2n} < A < S_{2n} + (S_{2n} - S_n)$ .... (6)

Also, $S_{2n} = n \times \frac{ra_n}{2} = \frac{1}{2}P_n r$ .... (7)

When $n = 96$, $r = 1$, $a_n = 0.065438$, $P_2 = 6.282048 = 2 \times 3.141024$

$S_{2n} = 3.139344$ $\quad s_{2n} + (S_{2n} - S_n) = 3.146064$

$\therefore\quad 3.139344 < \pi < 3.146064$

(iv) In Diagram 1, let $AB = a_{2^n}$, $BC = a_{2^{n+1}} = a_{2\times 2^n}$

From (7), $S_{2^n} = 2^n \times \frac{ra_{2^{n-1}}}{2}, S_{2^{n+1}} = 2^{n+1} \times \frac{ra_{2^n}}{2}$ .... (8)

Divide the two eq. in (*), $\frac{S_{2^n}}{S_{2^{n+1}}} = \frac{1}{2}\frac{a_{2^{n-1}}}{a_{2^n}} = \frac{\frac{1}{2}AB}{BC} = \frac{BD}{BC} = \cos\angle DBC = \cos\left(\frac{\pi}{2} - \angle DCB\right)$

$$= \cos\left(\frac{\pi}{2} - \frac{1}{2}(\pi - \angle BOC)\right) = \cos\frac{1}{2}\angle BOC = \cos\frac{1}{2}\frac{360^\circ}{2^n} = \cos\frac{180^\circ}{2^n}$$

(v) From (vi), $\frac{1}{S_{2^{n+1}}} = \frac{1}{S_{2^n}}\cos\frac{180^\circ}{2^n} = \frac{1}{S_{2^{n-1}}}\cos\frac{180^\circ}{2^{n-1}}\cos\frac{180^\circ}{2^n} = ...$

$$= \frac{1}{S_{2^2}}\cos\frac{180^\circ}{2^2}\cos\frac{180^\circ}{2^3}...\cos\frac{180^\circ}{2^{n-1}}\cos\frac{180^\circ}{2^n} = \frac{1}{2r^2}\cos 45^\circ\cos\frac{45^\circ}{2}\cos\frac{45^\circ}{4}...\cos\frac{45^\circ}{2^{n-2}}$$

Taking limit on both side as $n \to \infty$,

$$\frac{1}{\pi r^2} = \frac{1}{2r^2}\cos 45^\circ\cos\frac{45^\circ}{2}\cos\frac{45^\circ}{4}...$$

$$\therefore\quad \frac{2}{\pi} = \cos 45^\circ\cos\frac{45^\circ}{2}\cos\frac{45^\circ}{4}....$$

(vi) $\cos 45^\circ = \sqrt{\frac{1}{2}}$, $\cos\frac{45^\circ}{2} = \sqrt{\frac{1+\cos 45^\circ}{2}} = \sqrt{\frac{1}{2}+\left(1+\sqrt{\frac{1}{2}}\right)}$,

$$\cos\frac{45^\circ}{2^2} = \sqrt{\frac{1+\cos\frac{45^\circ}{2}}{2}} = \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\right)}, \ldots.$$

$$\pi = \frac{2}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\left(1+\sqrt{\frac{1}{2}}\right)}\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\right)}\ldots}$$

5. Let P(n) be the proposition : $X \setminus \left(\bigcap_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n}(X \setminus A_i)$ $\forall\, n \in \mathbf{N}$.

For P(1), L.H.S. = $X \setminus A_1$ = R.H.S. $\therefore$ P(1) is true.

Assume P(k) is true for some $n \in \mathbf{N}$. i.e. $X \setminus \left(\bigcap_{i=1}^{k} A_i\right) = \bigcup_{i=1}^{k}(X \setminus A_i)$ …. (*)

For P(k+1), L.H.S. = $X \setminus \left(\bigcap_{i=1}^{k+1} A_i\right) = X \setminus \left[\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right] = \left(X \setminus \bigcap_{i=1}^{k} A_i\right) \cup (X \setminus A_{k+1})$, by (*)

$= \left[\bigcup_{i=1}^{k}(X \setminus A_i)\right] \cup (X \setminus A_{k+1}) = \bigcup_{i=1}^{k+1}(X \setminus A_i)$ = R.H.S. $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.

6. Let P(n) be the proposition : $\frac{d^n}{dx^n}\left(\frac{1}{1+x}\right) = \frac{n!(-1)^n}{(1+x)^{n+1}}$

For P(1), $\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$ $\therefore$ P(1) is true.

Assume P(k) is true for some $n \in \mathbf{N}$. i.e. $\frac{d^k}{dx^k}\left(\frac{1}{1+x}\right) = \frac{k!(-1)^k}{(1+x)^{k+1}}$ …. (*)

For P(k+1), $\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(\frac{1}{1+x}\right)\right] = \frac{d}{dx}\frac{k!(-1)^k}{(1+x)^{k+1}}$, by (*)

$$= \frac{0 - k!(-1)^k \frac{d}{dx}(1+x)^{k+1}}{(1+x)^{2(k+1)}} = \frac{-k!(-1)^k(k+1)(1+x)^k}{(1+x)^{2(k+1)}} = \frac{(k+1)!(-1)^{k+1}}{(1+x)^{k+2}}$$ $\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.